Canonical Statistics of Trapped Ideal and Interacting Bose Gases

Hongwei Xiong[1,2], Shujuan Liu[1], Guoxiang Huang[3], and Zaixin Xu[3]
[1]*Department of Applied Physics, Zhejiang University of Technology, Hangzhou, 310032, China*
[2]*Zhijiang College, Zhejiang University of Technology, Hangzhou, 310012, China*
[3]*Department of Physics, East China Normal University, Shanghai, 200062, China*

(February 1, 2008)

The mean ground state occupation number and condensate fluctuations of interacting and non-interacting Bose gases confined in a harmonic trap are considered by using a canonical ensemble approach. To obtain the mean ground state occupation number and the condensate fluctuations, an analytical description for the probability distribution function of the condensate is provided directly starting from the analysis of the partition function of the system. For the ideal Bose gas, the probability distribution function is found to be a Gaussian one for the case of the harmonic trap. For the interacting Bose gas, using a unified approach the condensate fluctuations are calculated based on the lowest-order perturbation method and on Bogoliubov theory. It is found that the condensate fluctuations based on the lowest-order perturbation theory follow the law $\left\langle \delta^2 N_{\mathbf{0}} \right\rangle \sim N$, while the fluctuations based on Bogoliubov theory behave as $N^{4/3}$.

I. INTRODUCTION

The experimental achievement of Bose-Einstein condensation (BEC) in dilute alkali atoms [1], spin-polarized hydrogen [2] and recently in metastable helium [3] has enormously stimulated the theoretical research [4–6] on the ultracold bosons. Among the several intriguing questions on the statistical properties of trapped interacting Bose gases, the problem of condensate fluctuations $\left\langle \delta^2 N_{\mathbf{0}} \right\rangle$ of the mean ground state occupation number $\left\langle N_{\mathbf{0}} \right\rangle$ is of central importance. Apart from the intrinsic theoretical interest, it is foreseeable that such fluctuations will become experimentally testable in the near future [7]. On the other hand, the calculations of $\left\langle \delta^2 N_{\mathbf{0}} \right\rangle$ are crucial to investigate the phase collapse time of the condensate [8,9].

It is well known that within a grand canonical ensemble the fluctuations of the condensate are given by $\left\langle \delta^2 N_{\mathbf{0}} \right\rangle = N_{\mathbf{0}} \left(N_{\mathbf{0}} + 1 \right)$, implying that $\delta N_{\mathbf{0}}$ becomes of order N when the temperature approaches zero. To avoid this sort of unphysically large condensate fluctuations, a canonical (or a microcanonical) ensemble has to be used to investigate the fluctuations of the condensate. On the other hand, because in the experiment the trapped atoms are cooled continuously from the surrounding, the system can be taken as being in contact with a heat bath but the total number of particles in the system is conserved. Thus it is necessary to use the canonical ensemble to investigate the statistical properties of the trapped weakly interacting Bose gas.

Within the canonical as well as the microcanonical ensembles, the condensate fluctuations have been studied systematically in the case of an ideal Bose gas in a box [10–14], and in the presence of a harmonic trap [14–21]. Recently, the question of how interatomic interactions affect the condensate fluctuations has been an object of several theoretical investigations [22–27]. Idziaszek *et al.* [23] investigated the condensate fluctuations of interacting Bose gases using the lowest-order perturbation theory and a two-gas model, while Giorgini *et al.* [22] addressed this problem within a traditional particle-number-nonconserving Bogoliubov approach. Recently, Kocharovsky *et al.* [26] supported and extended the results of the work of Giorgini *et al.* [22] using a particle-number-conserving operator formalism.

Although the condensate fluctuations are thoroughly investigated in Ref. [22–26], to best our knowledge up to now an analytical description of the probability distribution function for the interacting Bose gas directly from the microscopic statistics of the system has not been given. Note that as soon as the probability distribution function of the system is obtained, it is straightforward to get the mean ground state occupation number and the condensate fluctuations. The purpose of the present work is an attempt to provide such an analytical description of the probability distribution function of interacting and non-interacting Bose gases based on the analysis of the partition function of the system.

We shall investigate in this paper the condensate fluctuations of interacting and non-interacting Bose gases confined in a harmonic trap. The analytical probability distribution function of the condensate will be given directly from the partition function of the system using a canonical ensemble approach. For an ideal Bose gas, we find that the probability distribution of the condensate is a Gaussian function. In particular, our method can be easily extended to discuss the probability distribution function for a weakly interacting Bose gas. A unified way is given to calculate the condensate fluctuations from the lowest-order perturbation theory and from Bogoliubov theory. We find that different methods of approximation for the interacting Bose gas give quite different predictions concerning the condensate fluctuations. We show

that the fluctuations based on the lowest-order perturbation theory follow the law $\langle \delta^2 N_{\mathbf{0}} \rangle \sim N$, while the fluctuations based on the Bogoliubov theory behave as $N^{4/3}$.

The paper is organized as follows. Sec. II is devoted to outline the canonical ensemble, which is developed to discuss the probability distribution function of Bose gases. In Sec. III we investigate the condensate fluctuations of the ideal Bose gas confined in a harmonic trap. In Sec. IV the condensate fluctuations of the weakly interacting Bose gas are calculated based on the lowest order perturbation theory. In Sec. V the condensate fluctuations due to collective excitations are obtained based on Bogoliubov theory. Finally, Sec. VI contains a discussion and summary of our results.

II. FLUCTUATIONS AND MEAN GROUND STATE OCCUPATION NUMBER OF THE CONDENSATE IN THE CANONICAL ENSEMBLE

According to the canonical ensemble, the partition function of the system with N trapped interacting bosons is given by

$$Z[N] = \sum_{\Sigma_{\mathbf{n}} N_{\mathbf{n}} = N} \exp\left[-\beta \left(\Sigma_{\mathbf{n}} N_{\mathbf{n}} \varepsilon_{\mathbf{n}} + E_{int}\right)\right], \qquad (1)$$

where $N_{\mathbf{n}}$ and $\varepsilon_{\mathbf{n}}$ are occupation number and energy level of the state $\mathbf{n} = \{n_x, n_y, n_z\}$, respectively. $\beta = 1/k_B T$ and $\{n_x, n_y, n_z\}$ are non-negative integers. E_{int} is the interaction energy of the system. For convenience, by separating out the ground state $\mathbf{n} = \mathbf{0}$ from the state $\mathbf{n} \neq \mathbf{0}$, we have

$$Z[N] = \sum_{N_{\mathbf{0}}=0}^{N} \{\exp\left[-\beta \left(E_{\mathbf{0}} + E_{int}\right)\right] Z_0(N, N_{\mathbf{0}})\}, \qquad (2)$$

where $Z_0(N, N_{\mathbf{0}})$ stands for the partition function of a fictitious system comprising $N - N_{\mathbf{0}}$ trapped ideal non-condensed bosons:

$$Z_0(N, N_{\mathbf{0}}) = \sum_{\sum_{\mathbf{n}\neq\mathbf{0}} N_{\mathbf{n}} = N - N_{\mathbf{0}}} \exp\left[-\beta \sum_{\mathbf{n}\neq\mathbf{0}} N_{\mathbf{n}} \varepsilon_{\mathbf{n}}\right]. \qquad (3)$$

Assuming $A_0(N, N_{\mathbf{0}})$ is the free energy of the fictitious system, we have

$$A_0(N, N_{\mathbf{0}}) = -k_B T \ln Z_0(N, N_{\mathbf{0}}). \qquad (4)$$

The calculation of the free energy $A_0(N, N_{\mathbf{0}})$ is nontrivial because there is a requirement that the number of non-condensed bosons is $N - N_{\mathbf{0}}$ in the summation of the partition function $Z_0(N, N_{\mathbf{0}})$. Using the saddle-point method developed by Darwin and Fowler [28], it is straightforward to obtain a useful relation between the free energy $A_0(N, N_{\mathbf{0}})$ and the fugacity z_0 of the fictitious $N - N_{\mathbf{0}}$ non-interacting bosons

$$-\beta \frac{\partial}{\partial N_{\mathbf{0}}} A_0(N, N_{\mathbf{0}}) = \ln z_0, \qquad (5)$$

where the fugacity z_0 is determined by

$$N_{\mathbf{0}} = N - \sum_{\mathbf{n}\neq\mathbf{0}} \frac{1}{\exp\left[\beta \varepsilon_{\mathbf{n}}\right] z_0^{-1} - 1}. \qquad (6)$$

We have given a simple derivation of Eqs. (5) and (6) in the Appendix.

Using the free energy $A_0(N, N_{\mathbf{0}})$, the partition function of the system becomes

$$Z[N] = \sum_{N_{\mathbf{0}}=0}^{N} \exp\left[q(N, N_{\mathbf{0}})\right], \qquad (7)$$

where

$$q(N, N_{\mathbf{0}}) = -\beta \left(E_{\mathbf{0}} + E_{int}\right) - \beta A_0(N, N_{\mathbf{0}}). \qquad (8)$$

It is obvious that $(1/Z[N]) \exp\left[q(N, N_{\mathbf{0}})\right]$ represents the probability finding $N_{\mathbf{0}}$ atoms in the condensate.

To obtain the probability distribution function of the system, let us first investigate the largest term in the sum of the partition function $Z[N]$. Assume the number of the condensed atoms is $N_{\mathbf{0}}^p$ in the largest term of the partition function. The largest term $\exp\left[q(N, N_{\mathbf{0}}^p)\right]$ is determined by requiring that $\frac{\partial}{\partial N_{\mathbf{0}}} q(N, N_{\mathbf{0}})|_{N_{\mathbf{0}} = N_{\mathbf{0}}^p} = 0$, *i.e.*,

$$-\beta \frac{\partial}{\partial N_{\mathbf{0}}^p} \left(E_{\mathbf{0}} + E_{int}\right) - \beta \frac{\partial}{\partial N_{\mathbf{0}}^p} A_0(N, N_{\mathbf{0}}^p) = 0. \qquad (9)$$

Using Eq. (5) we obtain

$$\ln z_0^p = \beta \frac{\partial}{\partial N_{\mathbf{0}}^p} \left(E_{\mathbf{0}} + E_{int}\right). \qquad (10)$$

In addition, from Eq. (6), the most probable value $N_{\mathbf{0}}^p$ is determined by

$$N_{\mathbf{0}}^p = N - \sum_{\mathbf{n}\neq\mathbf{0}} \frac{1}{\exp\left[\beta \varepsilon_{\mathbf{n}}\right] \left(z_0^p\right)^{-1} - 1}. \qquad (11)$$

In the case of an ideal Bose gas, from Eq. (10) one obtains $\ln z_0^p = \beta \varepsilon_{\mathbf{0}}$. Thus $N_{\mathbf{0}}^p$ is the same as the mean ground state occupation number obtained by using a grand canonical ensemble approach. For sufficiently large N, the sum $\sum_{N_{\mathbf{0}}=0}^{N}$ in (7) may be replaced by the largest term, since the error omitted in doing so is statistically negligible. In this situation, Eq. (11) shows the equivalence between the canonical ensemble and the grand canonical ensemble for large N.

The other terms in the partition function (7) will contribute to the fluctuations of the condensate, and lead to

the deviation of $\langle N_{\mathbf{0}} \rangle$ from the most probable value $N_{\mathbf{0}}^p$. If $N_{\mathbf{0}} \neq N_{\mathbf{0}}^p$, we have $\frac{\partial}{\partial N_{\mathbf{0}}} q(N, N_{\mathbf{0}}) \neq 0$. Assuming

$$\frac{\partial}{\partial N_{\mathbf{0}}} q(N, N_{\mathbf{0}}) = \alpha(N, N_{\mathbf{0}}), \tag{12}$$

from Eqs. (5) and (8), we obtain

$$\ln z_0 = \beta \frac{\partial}{\partial N_{\mathbf{0}}} (E_{\mathbf{0}} + E_{int}) + \alpha(N, N_{\mathbf{0}}). \tag{13}$$

By Eqs. (6) and (13), we have

$$N_{\mathbf{0}} = N - \sum_{\mathbf{n} \neq \mathbf{0}} \frac{1}{\exp[\beta \varepsilon_{\mathbf{n}}] \exp[-\beta \frac{\partial}{\partial N_{\mathbf{0}}} (E_{\mathbf{0}} + E_{int}) - \alpha(N, N_{\mathbf{0}})] - 1}. \tag{14}$$

Combining Eqs. (11) and (14), we get the following equation for determining $\alpha(N, N_{\mathbf{0}})$

$$N_{\mathbf{0}} - N_{\mathbf{0}}^p = \sum_{\mathbf{n} \neq \mathbf{0}} \frac{1}{\exp[\beta \varepsilon_{\mathbf{n}}] \exp[-\beta \frac{\partial}{\partial N_{\mathbf{0}}^p} (E_{\mathbf{0}} + E_{int})] - 1} - \sum_{\mathbf{n} \neq \mathbf{0}} \frac{1}{\exp[\beta \varepsilon_{\mathbf{n}}] \exp[-\beta \frac{\partial}{\partial N_{\mathbf{0}}} (E_{\mathbf{0}} + E_{int}) - \alpha(N, N_{\mathbf{0}})] - 1}. \tag{15}$$

Once we know $E_{\mathbf{0}}$ and E_{int} of the system, it is straightforward to obtain $\alpha(N, N_{\mathbf{0}})$ from Eq. (15). Using $\alpha(N, N_{\mathbf{0}})$, one can obtain the probability distribution function of the system.

From Eq. (12), we obtain the following result for $q(N, N_{\mathbf{0}})$

$$q(N, N_{\mathbf{0}}) = \int_{N_{\mathbf{0}}^p}^{N_{\mathbf{0}}} \alpha(N, N_{\mathbf{0}}) \, dN_{\mathbf{0}} + q(N, N_{\mathbf{0}}^p). \tag{16}$$

Thus the partition function of the system becomes

$$Z[N] = \sum_{N_{\mathbf{0}}=0}^{N} \{\exp[q(N, N_{\mathbf{0}}^p)] \, G(N, N_{\mathbf{0}})\}, \tag{17}$$

where

$$G(N, N_{\mathbf{0}}) = \exp\left[\int_{N_{\mathbf{0}}^p}^{N_{\mathbf{0}}} \alpha(N, N_{\mathbf{0}}) \, dN_{\mathbf{0}}\right]. \tag{18}$$

Assuming $P(N_{\mathbf{0}}|N)$ is the probability to find $N_{\mathbf{0}}$ atoms in the condensate, $G(N, N_{\mathbf{0}})$ represents the ratio $\frac{P(N_{\mathbf{0}}|N)}{P(N_{\mathbf{0}}^p|N)}$, *i.e.*, the relative probability to find $N_{\mathbf{0}}$ atoms in the condensate. From Eq. (18), the normalized probability distribution function is given by

$$G_n(N, N_{\mathbf{0}}) = A \exp\left[\int_{N_{\mathbf{0}}^p}^{N_{\mathbf{0}}} \alpha(N, N_{\mathbf{0}}) \, dN_{\mathbf{0}}\right], \tag{19}$$

where A is a normalization constant and is given by the condition $A \int G(N, N_{\mathbf{0}}) dN_{\mathbf{0}} = 1$.

As soon as we know $G(N, N_{\mathbf{0}})$, the statistical properties of the system can be clearly described. From Eqs. (17) and (18) one obtains the mean ground state occupation number $\langle N_{\mathbf{0}} \rangle$ and fluctuations $\langle \delta^2 N_{\mathbf{0}} \rangle$ in the canonical ensemble:

$$\langle N_{\mathbf{0}} \rangle = \frac{\sum_{N_{\mathbf{0}}=0}^{N} N_{\mathbf{0}} \exp[q(N, N_{\mathbf{0}})]}{\sum_{N_{\mathbf{0}}=0}^{N} \exp[q(N, N_{\mathbf{0}})]} = \frac{\sum_{N_{\mathbf{0}}=0}^{N} N_{\mathbf{0}} G(N, N_{\mathbf{0}})}{\sum_{N_{\mathbf{0}}=0}^{N} G(N, N_{\mathbf{0}})} \tag{20}$$

$$\langle \delta^2 N_{\mathbf{0}} \rangle = \langle N_{\mathbf{0}}^2 \rangle - \langle N_{\mathbf{0}} \rangle^2 = \frac{\sum_{N_{\mathbf{0}}=0}^{N} N_{\mathbf{0}}^2 G(N, N_{\mathbf{0}})}{\sum_{N_{\mathbf{0}}=0}^{N} G(N, N_{\mathbf{0}})} - \left[\frac{\sum_{N_{\mathbf{0}}=0}^{N} N_{\mathbf{0}} G(N, N_{\mathbf{0}})}{\sum_{N_{\mathbf{0}}=0}^{N} G(N, N_{\mathbf{0}})}\right]^2. \tag{21}$$

Starting from Eqs. (20) and (21), one can calculate the mean ground state occupation number and fluctuations for ideal and interacting Bose gases.

III. IDEAL BOSE GASES

We now study the condensate fluctuations of the system with N non-interacting bosons trapped in an external potential. The potential is a harmonic one with the form

$$V_{ext}(\mathbf{r}) = \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right), \tag{22}$$

where m is the mass of atoms, ω_x, ω_y, and ω_z are frequencies of the trap along three coordinate-axis directions. The single-particle energy level has the form

$$\varepsilon_{\mathbf{n}} = \left(n_x + \frac{1}{2}\right)\hbar\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\omega_y + \left(n_z + \frac{1}{2}\right)\hbar\omega_z. \tag{23}$$

From Eq. (11) one can get easily the most probable value $N_{\mathbf{0}}^p$, which reads

$$N_{\mathbf{0}}^p = N - N\left(\frac{T}{T_c^0}\right)^3 - \frac{3\overline{\omega}\zeta(2)}{2\omega_{ho}[\zeta(3)]^{2/3}}\left(\frac{T}{T_c^0}\right)^2 N^{2/3}, \tag{24}$$

where $T_c^0 = \frac{\hbar\omega_{ho}}{k_B}\left(\frac{N}{\zeta(3)}\right)^{1/3}$ is the critical temperature of the ideal Bose gas in the thermodynamic limit. $\overline{\omega} = (\omega_x + \omega_y + \omega_z)/3$ and $\omega_{ho} = (\omega_x\omega_y\omega_z)^{1/3}$ are arithmetic

$\left\langle \delta^2 N_{\mathbf{0}}\right\rangle \sim N^{4/3}$. In this section, the Bogoliubov theory will be developed based on our canonical statistics to discuss the condensate fluctuations originating from collective excitations. According to the Bogoliubov theory [39,40], the total number of particles out of the condensate is given by

$$N_T = \sum_{nl\neq 0} N_{nl} = \sum_{nl\neq 0} (u_{nl}^2 + v_{nl}^2) f_{nl}. \tag{47}$$

The real quantities u_{nl} and v_{nl} satisfy the relations

$$u_{nl}^2 + v_{nl}^2 = \frac{\left[\left(\varepsilon_{nl}^B\right)^2 + g^2 n_0^2\right]^{1/2}}{2\varepsilon_{nl}^B}, \tag{48}$$

$$u_{nl} v_{nl} = -\frac{g n_0}{2\varepsilon_{nl}^B}, \tag{49}$$

where f_{nl} is the number of the collective excitations excited in the system at the thermal equilibrium

$$f_{nl} = \frac{1}{\exp\left[\beta \varepsilon_{nl}^B\right] - 1}. \tag{50}$$

In addition, the energy of the collective excitations entering Eqs. (48) and (49) is given by the dispersion law [41]

$$\varepsilon_{nl}^B = \hbar\omega_{ho}\left(2n^2 + 2nl + 3n + l\right)^{1/2}. \tag{51}$$

These phonon-like collective excitations are in excellent agreement with the measurement of experiments. The dispersion law (51) is valid if the conditions $N_{\mathbf{0}} a/a_{ho} >> 1$ and $\varepsilon_{nl} << \mu$ are satisfied. The contribution to the condensate fluctuations due to these discrete low energy modes are important because $f_{nl}, u_{nl}^2 + v_{nl}^2, u_{nl}v_{nl} \propto 1/\sqrt{2n^2 + 2nl + 3n + l}$ at low excitation energies.

In Eq. (47) N_{nl} can be regarded as the effective occupation number of non-condensed atoms, while

$$N_{nl}^B = \frac{N_{nl}}{u_{nl}^2 + v_{nl}^2} = f_{nl} \tag{52}$$

is the occupation number of the collective excitations. From the form of f_{nl} one can construct the partition function of the collective excitations in the frame of canonical ensemble

$$Z_B = \sum_{\{nl\}} \exp\left[-\beta \sum_{nl} N_{nl}^B \varepsilon_{nl}^B\right]. \tag{53}$$

From Eq. (52) Z_B becomes

$$Z_B = \sum_{\{\Sigma N_{nl} = N\}} \exp\left[-\beta \sum_{nl} N_{nl} \varepsilon_{nl}^{eff}\right], \tag{54}$$

where $\varepsilon_{nl}^{eff} = \varepsilon_{nl}^B / \left(u_{nl}^2 + v_{nl}^2\right)$ can be taken as an effective energy level of the thermal atoms. In this case Z_B is the partition function of a fictitious boson system, which is composed of N non-interacting Bosons whose energy level is determined by ε_{nl}^{eff}. From (54) the most probable value $N_{\mathbf{0}}^p$ is given by

$$N_{\mathbf{0}}^p = N - \sum_{nl\neq 0} \frac{1}{\exp\left[\beta\left(\varepsilon_{nl}^{eff} - \varepsilon_{nl=0}^{eff}\right)\right] - 1}. \tag{55}$$

It is obvious that the occupation number of low n, l in Eq. (55) coincides with that of Eq. (47). Other $N_{\mathbf{0}}$ is determined by

$$N_{\mathbf{0}} = N - \sum_{nl\neq 0} \frac{1}{\exp\left[\beta\left(\varepsilon_{nl}^{eff} - \varepsilon_{nl=0}^{eff}\right)\right] \exp\left[-\alpha\left(N, N_{\mathbf{0}}\right)\right] - 1}. \tag{56}$$

From Eqs. (55) and (56) we obtain

$$\alpha\left(N, N_{\mathbf{0}}\right) \approx -\frac{N_{\mathbf{0}} - N_{\mathbf{0}}^p}{\sum_{nl\neq 0}\left(u_{nl}^2 + v_{nl}^2\right)^2 f_{nl}^2}. \tag{57}$$

When getting (57) we have used the approximation $f_{nl} \approx k_B T/\varepsilon_{nl}^B$ for low energy collective excitations. Thus the probability distribution function of the condensate is given by

$$G_B\left(N, N_{\mathbf{0}}\right) = A_B \exp\left[-\frac{\left(N_{\mathbf{0}} - N_{\mathbf{0}}^p\right)^2}{2\sum_{nl\neq 0}\left(u_{nl}^2 + v_{nl}^2\right)^2 f_{nl}^2}\right], \tag{58}$$

where A_B is a normalization constant. Therefore, the condensate fluctuations due to the collective excitations reads

$$\left\langle \delta^2 N_{\mathbf{0}}\right\rangle_{collective} = \frac{\sum_{N_{\mathbf{0}}=0}^{N} N_{\mathbf{0}}^2 G_B\left(N, N_{\mathbf{0}}\right)}{\sum_{N_{\mathbf{0}}=0}^{N} G_B\left(N, N_{\mathbf{0}}\right)} - \left[\frac{\sum_{N_{\mathbf{0}}=0}^{N} N_{\mathbf{0}} G_B\left(N, N_{\mathbf{0}}\right)}{\sum_{N_{\mathbf{0}}=0}^{N} G_B\left(N, N_{\mathbf{0}}\right)}\right]^2. \tag{59}$$

Eqs. (58) and (59) provide the formulas for calculating the condensate fluctuations originating from the collective excitations.

Below the temperature T_m which corresponds to the maximum fluctuations, we obtain the analytical result for the condensate fluctuations

$$\left\langle \delta^2 N_0\right\rangle_{collective} = \frac{\pi^2}{12\zeta\left(2\right)} B\left(\frac{m a^2 k_B T_c}{\hbar^2}\right)^{2/5} N^{4/3} = \frac{1}{2} B\left(\frac{m a^2 k_B T_c}{\hbar^2}\right)^{2/5} N^{4/3}, \tag{60}$$

where B is a dimensionless parameter, which is the same as that obtained in Ref. [22]. Note that compared with the result obtained by Ref. [22], the coefficient in (60) differs by a factor $\frac{1}{2}$. The expression (60) shows clearly that the condensate fluctuations due to the collective excitations are anomalous, *i.e.*, proportional to $N^{4/3}$. Note that $G_B\left(N, N_{\mathbf{0}}\right)$ is a Gaussian distribution function, the anomalous behavior of the condensate fluctuations comes from the factor $2\sum_{nl\neq 0}\left(u_{nl}^2+v_{nl}^2\right)^2 f_{nl}^2$, which is proportional to $N^{4/3}$.

At the critical temperature the probability distribution is given by $G_B\left(T=T_c\right) = \exp\left[-N_{\mathbf{0}}^2/\gamma\right]$, where $\gamma = 2\sum_{nl}\left(u_{nl}^2+v_{nl}^2\right)^2 f_{nl}^2$. In this case, we obtains the analytical result of the condensate fluctuations

$$\left\langle \delta^2 N_0\right\rangle|_{T=T_c} = 0.18\gamma = 0.18B\left(\frac{ma^2 k_B T_c}{\hbar^2}\right)^{2/5} N^{4/3}. \tag{61}$$

From (60) and (61) we find that the behavior of the condensate fluctuations based on the Bogoliubov theory is rather different from that of the lowest-order perturbation theory.

VI. DISCUSSION AND CONCLUSION

In this paper, a canonical ensemble approach has been developed to investigate the mean ground state occupation number and condensate fluctuations for interacting and non-interacting Bose gases. Different from the conventional methods, the analytical probability distribution function of the condensate has been obtained directly from the partition function of the system. Based on the probability distribution function, we have calculated the thermodynamic properties of the Bose gas, such as the condensate fraction and the fluctuations. Through the calculations of the probability distribution function, we have provided a simple method to recover the applicability of the saddle-point method for studying the condensate fluctuations. In fact, the theory of the improved saddle-point method developed in this work can be applied straightforwardly to consider the condensate fluctuations in other physical systems, such as the interacting Bose gas confined in a box [42], the interacting Bose gas in low-dimensions, etc.. The probability distribution function can also be used to discuss other interesting problems, such as the phase diffusion of the condensate.

For the harmonically trapped interacting Bose gas, we found that different approximations for weakly interacting Bose gases give quite different theoretical predictions concerning the condensate fluctuations. In our opinion the lowest-order perturbation theory gives in some sense the condensate fluctuations due to normal thermal atoms, while the Bogoliubov theory gives the condensate fluctuations originating from the collective excitations. The contributions to the condensate fluctuations due to the collective excitations mainly come from the low energy modes, and it is obvious that the condensate fluctuations based on the lowest-order perturbation theory miss the contributions coming from the collective excitations. Considering the fact that the contributions due to low energy thermal atoms in the lowest order perturbation theory is relatively small, the overall condensate fluctuations may be written as

$$\left\langle \delta^2 N_{\mathbf{0}}\right\rangle_{all} = \left\langle \delta^2 N_{\mathbf{0}}\right\rangle_{int} + \left\langle \delta^2 N_{\mathbf{0}}\right\rangle_{collective}, \tag{62}$$

where $\left\langle \delta^2 N_{\mathbf{0}}\right\rangle_{int}$ and $\left\langle \delta^2 N_{\mathbf{0}}\right\rangle_{collective}$ are condensate fluctuations due to the normal thermal atoms and the collective excitations, respectively.

ACKNOWLEDGMENT

This work was supported by the Science Foundation of Zhijiang College, Zhejiang University of Technology and Natural Science Foundation of Zhejiang Province. G. X. Huang was supported by the National Natural Science Foundation of China, the Trans-Century Training Programme Foundation for the Talents and the University Key Teacher Foundation of Chinese Ministry of Education. S. J. Liu and H. W. Xiong thank Professors G. S. Jia and J. F. Shen for their enormous encouragement.

APPENDIX

In this appendix, the method of saddle-point integration described by Darwin and Fowler [28] is used to investigate the partition function of the fictitious $N-N_{\mathbf{0}}$ non-interacting bosons. The partition function of the fictitious system is given by

$$Z_0\left(N_T\right) = \sum_{\sum_{\mathbf{n}\neq\mathbf{0}} N_{\mathbf{n}}=N_T} \exp\left[-\beta\sum_{\mathbf{n}\neq\mathbf{0}} N_{\mathbf{n}}\varepsilon_{\mathbf{n}}\right], \tag{63}$$

where $N_T = N - N_{\mathbf{0}}$ is the number of particles out of the condensate.

Because of the restriction $\sum_{\mathbf{n}\neq\mathbf{0}} N_{\mathbf{n}} = N_T$ in the summation of Eq. (63), $Z_0\left(N_T\right)$ can not be explicitly evaluated. To proceed we define a generating function for $Z_0\left(N_T\right)$ in the following manner. For any complex number z, we take

$$G_0\left(T,z\right) = \sum_{N_T=0}^{\infty} z^{N_T} Z_0\left(N_T\right). \tag{64}$$

The generating function can be evaluated easily. The result of $G_0\left(T,z\right)$ is given by

$$G_0\left(T,z\right) = \prod_{\mathbf{n}\neq\mathbf{0}} \frac{1}{1-z\exp\left[-\beta\varepsilon_{\mathbf{n}}\right]}. \tag{65}$$

To obtain $Z_0(N_T)$ we note that by definition $Z_0(N_T)$ is the coefficient of z^{N_T} in the expansion of $G_0(T,z)$ in powers of z. Therefore we have

$$Z_0(N_T) = \frac{1}{2\pi i}\oint dz \frac{G_0(T,z)}{z^{N_T+1}}, \tag{66}$$

where the contour of integration is a closed path in the complex z plane about $z=0$. Let $g(z)$ be defined by

$$\exp[g(z)] = \frac{G_0(T,z)}{z^{N_T+1}}, \tag{67}$$

then $Z_0(N_T)$ becomes

$$Z_0(N_T) = \frac{1}{2\pi i}\oint dz \exp[g(z)]. \tag{68}$$

The saddle point z_0 is determined by

$$\frac{\partial g(z_0)}{\partial z_0} = 0. \tag{69}$$

From Eq. (67) we obtain

$$N_T = z_0 \frac{\partial}{\partial z_0} \ln G_0(T,z_0) - 1. \tag{70}$$

By Eq. (65), one gets

$$N_T = \sum_{\mathbf{n}\neq \mathbf{0}} \frac{1}{\exp[\beta\varepsilon_{\mathbf{n}}] z_0^{-1} - 1}. \tag{71}$$

Noting that Eq. (71) is exactly the equation to determine the number of condensed atoms within the grand canonical ensemble, the saddle point z_0 can be also regarded as the fugacity of the fictitious $N - N_{\mathbf{0}}$ non-interacting bosons.

Expanding the integrand of Eq. (68) about $z = z_0$, we have

$$Z_0(N_T) = \frac{1}{2\pi i}\oint dz \exp\Big[g(z_0) + \frac{1}{2}(z-z_0)^2 \frac{\partial^2}{\partial z_0^2} g(z_0) + \cdots\Big], \tag{72}$$

where

$$\frac{\partial^2}{\partial z_0^2} g(z_0) = \frac{G_0''(T,z_0)}{G_0(T,z_0)} - \frac{N_T^2 - N_T}{z_0^2}. \tag{73}$$

By putting $z - z_0 = iy$ we obtain

$$Z_0(N_T) \approx \frac{\exp[g(z_0)]}{2\pi} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\frac{\partial^2}{\partial z_0^2} g(z_0) y^2\right] dy. \tag{74}$$

Thus we have

$$Z_0(N_T) = \frac{G_0(T,z_0)}{z_0^{N_T+1}[2\pi g''(z_0)]^{1/2}}. \tag{75}$$

With these results the free energy $A_0(N,N_{\mathbf{0}})$ of the fictitious system is then given by

$$A_0(N,N_{\mathbf{0}}) = -k_B T \Big\{\ln G_0(T,z_0) - N_T \ln z_0 - \ln z_0 - \frac{1}{2}\ln[2\pi g''(z_0)]\Big\}. \tag{76}$$

In the case of $N_T >> 1$, the last two terms in Eq. (76) can be omitted. Therefore

$$A_0(N,N_{\mathbf{0}}) = -k_B T[\ln G_0(T,z_0) - N_T \ln z_0]. \tag{77}$$

From Eq. (65), we obtain the relation between $A_0(N,N_{\mathbf{0}})$ and z_0 of the fictitious system:

$$-\beta \frac{\partial}{\partial N_{\mathbf{0}}} A_0(N,N_{\mathbf{0}}) = \ln z_0. \tag{78}$$

Eqs. (71) and (78) are useful relations used in the text.

---

[1] M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman and E.A. Cornell, Science **269**, 198 (1995); K.B. Davis, M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn and W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995); C.C. Bradley, C.A. Sackett, J.J. Tollett and R.G. Hulet, Phys. Rev. Lett. **75**, 1687 (1995).

[2] T.C. Killian, D.G. Fried, L. Willmann, D. Landhuis, S.C. Moss, T.J. Greytak and D. Kleppner, Phys. Rev. Lett. **81**, 3807 (1998); D.G. Fried, T.C. Killian, L. Willmann, D. Landhuis, S.C. Moss, D. Kleppner and T.J. Greytak, Phys. Rev. Lett. **81**, 3811 (1988).

[3] A. Robert, O. Sirjean, A. Browaeys, J. Poupard, S. Nowak, D. Boiron, C.I. Westbrook, A. Aspect, Science **292**, 461 (2001); F. Pereira Dos Santos, J. Leonard, J. Wang, C.J. Barrelet, F. Perales, E. Rasel, C.S. Unnikrishnan, M. Leduc, C. Cohen-Tannoudji, Phy. Rev. Lett. **86**, 3459 (2001).

[4] F. Dalfovo, S. Giorgini, L.P. Pitaevskii and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).

[5] A.S. Parkin and D.F. Walls, Phys. Rep. **303**, 1 (1998).

[6] A.J. Leggett, Rev. Mod. Phys. **73**, 307 (2001).

[7] Z. Idziaszek, K. Rzażewski and M. Lewenstein, Phys. Rev. A **61**, 053608 (2000).

[8] J. Javanainen and S.M. Yoo, Phys. Rev. Lett. **76**, 161 (1996); J. Javanainen and M. Wilkens, Phys. Rev. Lett. **78**, 4675 (1997).

[9] E.M. Wright, D.F. Walls and J.C. Garrison, Phys. Rev. Lett. **77**, 2158 (1996); E.M. Wright, T. Wong, M.J. Collett, S.M. Tan and D.F. Walls, Phys. Rev. A **56**, 591 (1997).

[10] E.H. Hauge, Physica Norvegica **4**, 19 (1969).
[11] I. Fujiwara, D. ter Haar and H. Wergeland, J. Stat. Phys. **2**, 329 (1970).
[12] R.M. Ziff, G.E. Uhlenbeck and M. Kac, Phys. Rep. **32**, 169 (1977).
[13] P. Borrmann, J. Harting, O. Mülken and E.R. Hilf, Phys. Rev. A **60**, 1519 (1999); P. Borrmann and G. Franke, J. Chem. Phys. **98**, 2484 (1993).
[14] M. Wilkens and C. Weiss, J. Mod. Opt. **44**, 1801 (1997).
[15] H.D. Politzer, Phys. Rev. A **54**, 5048 (1996).
[16] M. Gajda and K. Rzażewski, Phys. Rev. Lett. **78**, 2686 (1997).
[17] S. Grossmann and M. Holthaus, Phys. Rev. Lett. **79**, 3557 (1997).
[18] P. Navez, D. Bitouk, M. Gajda, Z. Idziaszek and K. Rzażewski, Phys. Rev. Lett. **79**, 1789 (1997).
[19] N.L. Balazs and T. Bergeman, Phys. Rev. A **58**, 2359 (1998).
[20] M. Holthaus, E. Kalinowski and K. Kirsten, Ann. Phys. (N. Y.) **270**, 198 (1998).
[21] M. Holthaus, E. Kalinowski, Ann. Phys. (N. Y.) **276**, 321 (1999).
[22] S. Giorgini, L.P. Pitaevskii and S. Stringari, Phys. Rev. Lett. **80**, 5040 (1998).
[23] Z. Idziaszek, M. Gajda, P. Navez, M. Wilkens and K. Rzażewski, Phys. Rev. Lett. **82**, 4376 (1999).
[24] F. Illuminati, P. Navez and M. Wilkens, J. Phys. B **32**, L461 (1999).
[25] F. Meier and W. Zwerger, Phys. Rev. A **60**, 5133 (1999).
[26] V.V. Kocharovsky, Vl.V. Kocharovsky and M.O. Scully, Phys. Rev. Lett. **84**, 2306 (2000).
[27] D. Jaksch, C.W. Gardiner, K.M. Gheri, and P. Zoller, Phys. Rev. A **58**, 1450 (1998).
[28] C.G. Darwin and R.H. Fowler, Phil. Mag. **44**, 450, 823 (1922); Proc. Combridge Phil. Soc. **21**, 262 (1922).
[29] R.B. Dingle, Proc. Cambridge. Phil. Soc, **45** 275 (1949); A.R. Fraser, Phil. Mag. **42**, 165 (1951).
[30] S. Grossmann and M. Holthaus, Phys. Lett. A **208**, 188 (1995); W. Ketterle and N.J. van Druten, Phys. Rev. A **54**, 656 (1996); K. Kirsten and D.J. Toms, Phys. Rev. A **54**, 4188 (1996); T. Haugset, H. Haugerud and J.O. Andersen, Phys. Rev. A **55**, 2922 (1997).
[31] O. Mülken, P. Borrmann, J. Harting and H. Stamerjohanns, Phys. Rev. A **64**, 013611 (2001).
[32] J.E. Robinson, Phys. Rev. **83**, 678 (1951); F. London, *Superfluids* (Wiley, New York, 1954), Vol. II Appendix p.203.
[33] S. Giorgini, L.P. Pitaevskii and S. Stringari, Phys. Rev. A **54**, R4633 (1996).
[34] M. Naraschewski, D.M. Stamper-Kurn, Phys. Rev. A **58**, 2423 (1998).
[35] T. Bergeman, D.L. Feder, N.L. Balazs and B.I. Schneider, Phys. Rev. A **61**, 063605 (2000).
[36] S.J. Liu, G.X. Huang, L. Ma, S.H. Zhu and H.W. Xiong, J. Phys. B **33**, 3911 (2000).
[37] H.W. Xiong, S.J. Liu, G.X. Huang, Z.J. Xu and C.Y. Zhang, J. Phys. B **34**, 3013 (2001).
[38] J.R. Ensher, D.S. Jin, M.R. Matthews, C.E. Wieman and E.A. Cornell, Phys. Rev. Lett. **77**, 4984 (1996).
[39] N. Bogoliubov, J. Phys. USSR **11**, 23 (1947).
[40] S. Giorgini, L.P. Pitaevskii and S. Stringari, J. Low. Temp. Phys. **109**, 309 (1997).
[41] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996).
[42] H.W. Xiong, S.J. Liu, G.X. Huang, Z.J. Xu and C.Y. Zhang, cond-mat/0106363 (to be published in J. Phys. B)